THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH3070 Introduction to Topology 2017-2018
Solution of Tutorial Classwork 9

1. Suppose $\mathbb{S}^1$ is a retract of $\mathbb{D}^2$. Then there exists $r : \mathbb{D}^2 \to \mathbb{S}^1$ such that $r \circ i = \mathrm{id}_{\mathbb{S}^1}$. This implies that $r_\# \circ i_\# = \mathrm{id}_\# = \mathrm{id}$. Hence $r_\# : \pi_1(\mathbb{D}^2, x_0) \to \pi_1(\mathbb{S}^1, x_0)$ must be surjective. However, $\pi_1(\mathbb{D}^2, x_0) \simeq \{x_0\}$ and $\pi_1(\mathbb{S}^1, x_0) \simeq (\mathbb{Z}, +)$. It is impossible to have a surjection from $\{x_0\}$ to $\mathbb{Z}$. Hence $\mathbb{S}^1$ is not a retract of $\mathbb{D}^2$.

2. Suppose yes. Then we have $\pi_1(\mathbb{S}^1 \times \mathbb{S}^1) = \pi_1(\mathbb{S}^1)$. However, since $\pi_1(X \times Y) = \pi_1(X) \times \pi_1(Y)$, $\pi_1(\mathbb{S}^1 \times \mathbb{S}^1) = \pi_1(\mathbb{S}^1) \times \pi_1(\mathbb{S}^1) \simeq (\mathbb{Z} \oplus \mathbb{Z}, +) \not\simeq (\mathbb{Z}, +) \simeq \pi_1(\mathbb{S}^1)$. This leads to contradiction. Hence the circle $\mathbb{S}^1 \times \{1\}$ is not a deformation retract of the torus.

3. (a) Define a mapping $r : \mathbb{D}^n \backslash \{0\} \to \mathbb{S}^{n-1}$ by $r(x) = \dfrac{x}{|x|}$. Clearly we have $r|_{\mathbb{S}^{n-1}} = \mathrm{id}$. Consider the homotopy $H(x,t) : \mathbb{D}^n \times [0,1] \to \mathbb{D}^n$ defined by $H(x,t) = tx + (1-t)r(x)$. Clearly H is continuous and satisfies $H(x,0) = r(x)$, $H(x,1) = x$. This shows that $r \simeq \mathrm{id}_{\mathbb{D}^n}$.

   (b) * Suppose there exists a homeomorphism $f : \mathbb{D}^2 \to \mathbb{D}^n$ for some $n > 2$. WLOG assume that $f(0) = 0$. Then we have a homeomorphism $f : \mathbb{D}^2 \backslash \{0\} \to \mathbb{D}^n \backslash \{0\}$. This implies that $\pi_1(\mathbb{D}^2 \backslash \{0\}) = \pi_1(\mathbb{D}^n \backslash \{0\})$. By a), we have $\pi_1(\mathbb{S}^1) \simeq \pi_1(\mathbb{D}^2 \backslash \{0\}) = \pi_1(\mathbb{D}^n \backslash \{0\}) \simeq \pi_1(\mathbb{S}^{n-1})$. However, $\pi_1(\mathbb{S}^1) \simeq (\mathbb{Z}, +)$ and $\pi_1(\mathbb{S}^{n-1}) \simeq \{1\}$. This leads to contradiction. Hence $\mathbb{D}^2$ and $\mathbb{D}^n$ are not homeomorphic.